# Advantages of the Laplace transform approach in pricing first touch digital options in Lévy-driven models*

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6 December 2015

**Abstract**

Motivated by the pricing of first touch digital options in exponential Lévy models and corresponding credit risk applications, we study numerical methods for solving related partial integro-differential equations. The goal of the paper is to consider advantages of the Laplace transform-based approach in this context. In particular, we show that the computational efficiency of the numerical methods which start with the time discretization can be significantly enhanced (often, in several dozen of times) by means of the Laplace transform technique. As an additional result we provide a new Wiener-Hopf factorization formula which admits an efficient numerical realization by means of the Fast Fourier Transform. We propose two new efficient methods for pricing first touch digital options in wide classes of Lévy processes. Both methods are based on the numerical Laplace transform inversion formulae and a numerical Wiener-Hopf factorization. The first method uses the Gaver-Stehfest algorithm, the second one deals with the Post-Widder formula. We prove the advantages of the new methods in terms of accuracy and convergence by using numerical experiments.

**Keywords:** Jump processes, Factorization theory, Laplace transform, Computational methods, Mathematical finance

## 1 Introduction

In recent years more and more attention has been given to stochastic models of financial markets which depart from the traditional Black-Scholes model. At this moment a wide range of models

*This research was supported by Russian Foundation for Humanities grant 15-32-01390.

is available. One of the tractable empirical models are jump diffusions or, more generally, Lévy processes. For an introduction on these models applied to finance, we refer to [12].

In the paper, we study the problem of pricing of first touch digital options in exponential Lévy models. We concentrate on the one-dimensional case and consider the model case of a continuously monitored option with a barrier from below. The problem is also important for credit risk applications, where the default events are often modeled as barrier crossing events and intra-horizon risk applications (see e.g. [17], [3]).

Let $T, H$ be the maturity, barrier, and $S_t = He^{X_t}$ be the stock price under a chosen risk-neutral measure. The riskless rate $r$ is assumed to be constant. We consider a first touch digital with a barrier from below $H$ and and an expiration date $T$. The contract pays \$1, as a stock price $S_t$ for first time the crosses the barrier $H$. If up to the date $T$ the price does not cross the barrier $H$, the option becomes worthless.

Then the no-arbitrage price of the first touch digital option at time $t < T$ and $X_t = x > 0$ is given by

$$F(t, x) = E\left[e^{-r(T'-t)}\mathbf{1}_{T'\le T} \mid X_t = x\right], \tag{1}$$

where $T'$ is the first entrance time into $(0, H]$.

The problem (1) is closely related to calculating first-passage probabilities. If the riskless rate $r = 0$, then

$$P\left[T' \le T \mid X_t = x\right] = F(t, x). \tag{2}$$

In the latter case, the expectations $F(t, x)$ are taken under the physical probability measure.

Option valuation under Lévy processes has been dealt with by a host of researchers, therefore, an exhaustive list is virtually impossible. One can find up-to-date surveys of the state of the art in computational finance and detailed bibliographies for further reading in [11]. The large group of numerical methods for pricing path-dependent options starts with the reduction to a boundary problem for the generalization of the Black-Scholes equation (backward Kolmogorov equation, [7, 12]). Then the methods typically use either a time discretization (the method of horizontal lines), see e.g. [13, 19, 28, 31], or Laplace transform with respect to the time variable, e.g. [34, 32, 29, 26, 40].

In the former case, one obtains a sequence of the certain stationary boundary problems for integro-differential equations on the line. Problems of the sequence can be solved by using either a finite difference scheme (e.g. [13, 33, 30]), or using the Fast Wiener-Hopf factorization method (FWHF-method), [28]. Usually one needs at least several hundred steps in time to achieve good accuracy (within 0.5%–1%). The FWHF-method is based on an efficient approximation of the Wiener-Hopf factors in the exact formula for the solution and the Fast Fourier Transform algorithm. In contrast to finite difference methods where the application entails a detailed analysis of the underlying Lévy model, the FWHF-method deals with the characteristic exponent of the process. Numerical examples with the comparison of the FWHF-method and finite difference schemes are reported in [28]. The paper [18] provides a formal solution to the continuously monitored double-barrier option using Toeplitz operator theory.

The Laplace transform-based approach is applicable if the characteristic exponent of the underlying Lévy process is rational. The basic examples are the Brownian motion, Kou's model and its generalization, the Hyper-Exponential Jump-Diffusion model (HEJD). In this case, the Laplace transform is derived explicitly from the distribution of the first passage time by applying the Wiener-Hopf factorization method. Once the Laplace transform is calculated, one uses a suitable numerical Laplace inversion algorithm to recover the option price. If the set of the option prices is needed one should repeat the procedure separately for each initial spot price of the underlying.

The general formulae for the prices of the barrier options in the Laplace domain involve the double Fourier inversion (and one more integration needed to calculate the factors in the Wiener-Hopf factorization formula), and hence it is difficult to implement them in practice In the general case, one can also approximate the initial process by Kou's model or by an HEJD, and then use the Laplace transform method (see e.g. [14, 22]). However, the additional source of the errors appears.

The goal of the paper is to show the advantage of the Laplace transform-based approach in the context of the option pricing under general Lévy processes. The idea behind our approach is to transform the problem to the Laplace domain where the solution is relatively easy to obtain by using the Fast Wiener-Hopf factorization method (or a finite difference scheme). The methods developed in the paper, in contrast to the other Laplace transform methods described above, can be applicable for the characteristic exponent of the general form. We will show that the Laplace transformation technique significantly reduces the computational complexity of the FWHF-method as well as finite difference schemes. We provide the order of the convergence of the horizontal lines method for the generalized Black-Scholes equation in the case of first touch digital options, and give the acceleration of the convergence formula. As an additional result we provide a new Wiener-Hopf factorization formula which admits an efficient numerical realization by means of the Fast Fourier Transform.

The Laplace transform maps the generalized Black- Scholes equation with the appropriate boundary conditions into the one-dimensional problem on the half-line parametrically dependent on the transform parameter. In our first approach, we solve the problems obtained by using the FWHF-method at real positive values of the transform parameter specified by the Gaver-Stehfest algorithm. Then option prices are computed via the numerical inversion formula. The second new approach is based on the Post-Widder formula.

We prove the advantages of the new methods in terms of accuracy and convergence by using numerical experiments. The methods achieve good accuracy (within 0.5%–1%) in few milliseconds, even near the barrier. Hence, these methods can be effectively used for calibration of Lévy models to barrier options.

The rest of the paper is organized as follows: in Section 2 we list the necessary facts of the theory of Lévy processes. Section 3 reviews the Fast Wiener-Hopf factorization method developed in [28], suggests a new Wiener-Hopf factorization formula which admits an efficient numerical realization by means of the Fast Fourier Transform, and introduces two enhanced FWHF-methods based on the numerical Laplace transform inversion. In Section 4, we produce numerical examples, and compare the results obtained by different methods; Section 5 concludes.

# 2 Lévy models

## 2.1 Lévy processes: general definitions

A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see e.g. [39]). A Lévy process may have a Gaussian component and/or pure jump component. The latter is characterized by the density of jumps, which is called the Lévy density. A Lévy process $X_t$ can be completely specified by its characteristic exponent, $\psi$, definable from the equality $E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$ (we confine ourselves to the one-dimensional case).

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty} (1 - e^{i\xi y} + i\xi y \mathbf{1}_{|y|\le 1}) F(dy), \tag{3}$$

where $\sigma^2 \ge 0$ is the variance of the Gaussian component, and the Lévy measure $F(dy)$ satisfies

$$\int_{\mathbf{R}\setminus\{\mathbf{0}\}} \min\{1, y^2\} F(dy) < +\infty. \tag{4}$$

Assume that under a risk-neutral measure chosen by the market, the stock has the dynamics $S_t = e^{X_t}$, where $X_t$ is a certain Lévy process. Then we must have $E[e^{X_t}] < +\infty$, and, therefore, $\psi$ must admit the analytic continuation into a strip $\Im\xi \in (-1, 0)$ and continuous continuation into the closed strip $\Im\xi \in [-1, 0]$. Further, if the riskless rate, $r \ge 0$, is constant, then the following condition (the EMM-requirement) must hold (see e.g. [7]): $E[e^{X_t}] = e^{rt}$. Equivalently, we have the relation

$$r + \psi(-i) = 0, \tag{5}$$

which can be used to express the drift $\mu$ via the other parameters of the Lévy process:

$$\mu = r - \frac{\sigma^2}{2} + \int_{-\infty}^{+\infty} (1 - e^y + y\mathbf{1}_{|y|\le 1}) F(dy). \tag{6}$$

The infinitesimal generator of $X$, denote it $L$, is an integro-differential operator which acts as follows:

$$Lu(x) = \frac{\sigma^2}{2} u''(x) + \mu u'(x) + \int_{-\infty}^{+\infty} (u(x+y) - u(x) - y\mathbf{1}_{|y|\le 1} u'(x)) F(dy). \tag{7}$$

The infinitesimal generator $L$ also can be represented as a pseudo-differential operator (PDO) with the symbol $-\psi(\xi)$, i.e. $L = -\psi(D)$. Recall that a PDO $A = a(D)$ acts as follows:

$$Au(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} a(\xi) \hat{u}(\xi) d\xi, \tag{8}$$

where $\hat{u}$ is the Fourier transform of a function $u$:

$$\hat{u}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} u(x) dx. \tag{9}$$

Note that the inverse Fourier transform in (8) is defined in the classical sense only if the symbol $a(\xi)$ and function $\hat{u}(\xi)$ are sufficiently nice. In general, one defines the (inverse) Fourier transform by duality.

## 2.2 Regular Lévy processes of exponential type

Loosely speaking, a Lévy process $X$ is called a *Regular Lévy Process of Exponential type* (RLPE) if its Lévy density has a polynomial singularity at the origin and decays exponentially at the infinity (see [7]). An almost equivalent definition is: the characteristic exponent is analytic in a strip $\Im\xi \in (\lambda_-, \lambda_+)$, $\lambda_- < -1 < 0 < \lambda_+$, continuous up to the boundary of the strip, and admits the representation

$$\psi(\xi) = -i\mu\xi + \phi(\xi), \tag{10}$$

where $\phi(\xi)$ stabilizes to a positively homogeneous function at the infinity:

$$\phi(\xi) \sim c_{\pm}|\xi|^{\nu}, \quad \text{as } \Re\xi \to \pm\infty, \quad \text{in the strip } \Im\xi \in (\lambda_-, \lambda_+), \tag{11}$$

where $c_{\pm} > 0$. "Almost" means that the majority of classes of Lévy processes used in empirical studies of financial markets satisfy conditions of both definitions. These classes are: Brownian motion, Kou's model [24], Hyperbolic processes [15, 16], Normal Inverse Gaussian processes and their generalization [4, 5], and extended Koponen's family. In [23] a symmetric version was introduced; [6] gave a non-symmetric generalization; later a subclass of this model appeared under the name CGMY–model in [10], and [7] used the name KoBoL family. The important exception is Variance Gamma Processes (VGP) [35]. VGP satisfy the conditions of the first definition but not the second one, since the characteristic exponent behaves like $const \cdot \ln|\xi|$, as $\xi \to \infty$.

**Example 1.** The characteristic exponent of a pure jump KoBoL process [7] (a.k.a. CGMY model [10]) of order $\nu \in (0, 2), \nu \neq 1$ is given by

$$\psi(\xi) = -i\mu\xi + c\Gamma(-\nu)[\lambda_+^{\nu} - (\lambda_+ + i\xi)^{\nu} + (-\lambda_-)^{\nu} - (-\lambda_- - i\xi)^{\nu}], \tag{12}$$

where $c > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

**Example 2.** If Lévy density is given by exponential functions on negative and positive axis:

$$F(dy) = \mathbf{1}_{(-\infty;0)}(y)c_+\lambda_+e^{\lambda_+ y}dy + \mathbf{1}_{(0;+\infty)}(y)c_-(-\lambda_-)e^{\lambda_- y},$$

where $c_{\pm} \geq 0$ and $\lambda_- < -1 < 0 < \lambda_+$, then we obtain Kou model [24]. The characteristic exponent of the process is of the form

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi}. \tag{13}$$

It is easy to show that Kou model is a compound Poisson process with double exponentially distributed jumps. Denote $p \in [0; 1]$ the probability of the upward jumps, $\lambda$ the arrival rate of jump sizes. Then we have $c_+ = \lambda(1 - p)$, $c_- = \lambda p$, and positive and negative jump sizes have exponential distribution with intensity $-\lambda_-$ and $\lambda_+$, respectively.

## 2.3 The Wiener-Hopf factorization

There are several forms of the Wiener-Hopf factorization. The Wiener-Hopf factorization formula used in probability reads:

$$E[e^{i\xi X_T}] = E[e^{i\xi \bar{X}_T}]E[e^{i\xi \underline{X}_T}], \quad \forall\, \xi \in \mathbf{R}, \tag{14}$$

where $T \sim \operatorname{Exp} q$, and $\bar{X}_t = \sup_{0\leq s\leq t} X_s$ and $\underline{X}_t = \inf_{0\leq s\leq t} X_s$ are the supremum and infimum processes. Introducing the notation

$$\phi_q^+(\xi) = qE\left[\int_0^{\infty} e^{-qt}e^{i\xi\bar{X}_t}dt\right] = E\left[e^{i\xi\bar{X}_T}\right], \tag{15}$$

$$\phi_q^-(\xi) = qE\left[\int_0^{\infty} e^{-qt}e^{i\xi\underline{X}_t}dt\right] = E\left[e^{i\xi\underline{X}_T}\right] \tag{16}$$

we can write (14) as

$$\frac{q}{q + \psi(\xi)} = \phi_q^+(\xi)\phi_q^-(\xi). \tag{17}$$

The equation (17) is a special case of the Wiener-Hopf factorization of the symbol of a PDO. In applications to Lévy processes, the symbol is $q/(q+\psi(\xi))$, and the PDO is $\mathcal{E}_q := q/(q-L) = q(q+\psi(D))^{-1}$: the normalized resolvent of the process $X_t$ or, using the terminology of [8], the expected present value operator (EPV–operator) of the process $X_t$. The name is due to the observation that, for a stream $g(X_t)$,

$$\mathcal{E}_q g(x) = E\left[\int_0^{+\infty} qe^{-qt} g(X_t)dt \mid X_0 = x\right].$$

Introduce the following operators:

$$\mathcal{E}_q^{\pm} := \phi_q^{\pm}(D), \tag{18}$$

which also admit interpretation as the EPV–operators under supremum and infimum processes.

One of the basic observations in the theory of PDO is that the product of symbols corresponds to the product of operators. In our case, it follows from (17) that

$$\mathcal{E}_q = \mathcal{E}_q^{+}\mathcal{E}_q^{-} = \mathcal{E}_q^{-}\mathcal{E}_q^{+} \tag{19}$$

as operators in appropriate function spaces.

For a wide class of Lévy models $\mathcal{E}$ and $\mathcal{E}^{\pm}$ admit interpretation as expectation operators:

$$\mathcal{E}_q g(x) = \int_{-\infty}^{+\infty} g(x+y)P_q(y)dy, \quad \mathcal{E}_q^{\pm} g(x) = \int_{-\infty}^{+\infty} g(x+y)P_q^{\pm}(y)dy,$$

where $P_q(y)$, $P_q^{\pm}(y)$ are certain probability densities with

$$P_q^{\pm}(y) = 0, \ \forall \pm y < 0.$$

Moreover, characteristic functions of the distributions $P_q(y)$ and $P_q^{\pm}(y)$ are $q(q+\psi(\xi))^{-1}$ and $\phi_q^{\pm}(\xi)$, respectively.

## 2.4 The generalized Black-Scholes equation

The price of any derivative contract, $F(t, X_t)$, will satisfy the Feynman-Kac formula, that is to say

$$(\partial_t + L - r)F(t,x) = 0, \tag{20}$$

where $x$ denotes the (normalized) log-price, $t$ denotes the time, and $L$ is the infinitesimal generator (under risk-neutral measure).

For the sake of brevity, consider the first touch digital option with the barrier from below $H$, maturity $T$, on a non-dividend paying stock $S_t$. Therefore, for the one-state Lévy process $X_t = \ln(S_t/H)$ with the generator (7), the derivative price, $F(t, X_t)$, will satisfy the following partial integro-differential equation (or more general pseudo-differential equation) with the appropriate initial and boundary conditions. See details in [7, 12].

$$\begin{aligned}(\partial_t + L - r)F(t,x) &= 0, \quad t < T, x > 0, &(21)\\ F(T,x) &= 0, \quad x > 0 &(22)\\ F(t,x) &= 1, \quad t \le T,\ x \le 0, &(23)\end{aligned}$$

where $a_+ = \max\{a, 0\}$. In addition, $F$ must be bounded.

If the characteristic exponent $\psi$ is sufficiently regular (e.g. $X_t$ belongs to the class of RLPE), then the general technique of the theory of PDO can be applied to show that a bounded solution, which is continuous on $\operatorname{supp} V \subset (-\infty, T) \times (0, +\infty)$, is unique – see, e.g., [27].

# 3 Laplace transform in the context of the FWHF-method

## 3.1 Numerical Laplace transform inversion: an overview

The Laplace transform is one of the classical methods for solving partial (integro)-differential equations which maps the problem to a space where the solution is relatively easy to obtain. The corresponding solution is referred to as the solution in the Laplace domain. In our case, the original function can not be retrieved analytically via computing the Bromwich's integral. Hence, the numerical inversion is needed.

We refer the reader to [2] for a description of a general framework for numerical Laplace transform inversion that contains the optimized version of the one-dimensional Gaver-Stehfest method.

Recall that popular in computational finance the Gaver-Stehfest algorithm for inverting Laplace transforms is related to the Post-Widder inversion formula. If $f(\tau)$ is a function of a nonnegative real variable $\tau$ and the Laplace transform $\tilde{f}(\lambda) = \int_0^\infty e^{-\lambda\tau} f(\tau)\, d\tau$ is known, the approximate Post-Widder formula for $f(\tau)$ can be written as

$$f(\tau) = \lim_{N\to\infty} f_N(\tau); \tag{24}$$

$$f_N(\tau) := \frac{(-1)^{N-1}}{(N-1)!}\left(\frac{N}{\tau}\right)^N \tilde{f}^{(N-1)}\left(\frac{N}{\tau}\right), \tag{25}$$

where $\tilde{f}^{(N)}(\lambda)$ – $N$th derivative of the Laplace transform $\tilde{f}$ at $\lambda$. It is well known that the convergence $f_N(\tau)$ to $f(\tau)$ as $N \to \infty$ is slow (of order $N^{-1}$), so acceleration is needed. In order to enhance the accuracy, [1] suggests a linear combination of the terms, i.e.,

$$f_{N,m}(\tau) = \sum_{k=1}^{m} w(k,m) f_{Nk}(\tau), \tag{26}$$

$$w(k,m) = (-1)^{m-k}\frac{k^m}{k!(m-k)!}. \tag{27}$$

In this case, convergence $f_{N,m}(\tau)$ to $f(\tau)$ is of order $N^{-m}$.

The methods of numerical Laplace inversion that fit the framework of [2] have the following general feature: the approximate formula for $f(\tau)$ can be written as

$$f(\tau) \approx \frac{1}{\tau}\sum_{k=1}^{N} \omega_k \cdot \tilde{f}\left(\frac{\alpha_k}{\tau}\right), \qquad 0 < \tau < \infty, \tag{28}$$

where $N$ is a positive integer and $\alpha_k$, $\omega_k$ are certain constants that are called the *nodes* and the *weights*, respectively. They depend on $N$, but not on $f$ or on $\tau$. In particular, the inversion formula of the Gaver-Stehfest method can written in the form (28) with

$$N = 2n; \tag{29}$$

$$\alpha_k = k\ln(2) \tag{30}$$

$$\omega_k := \frac{(-1)^{n+k}\ln(2)}{n!}\sum_{j=[(k+1)/2)]}^{\min\{k,n\}} j^{n+1} C_n^j C_{2j}^j C_j^{k-j}, \tag{31}$$

where $[x]$ is the greatest integer less than or equal to $x$ and $C_L^K = \frac{L!}{(L-K)!K!}$ are the binomial coefficients. Because of the binomial coefficients in the weights, the Gaver-Stehfest algorithm tends to require high system precision in order to yield good accuracy in the calculations.

From [41], we conclude that the required system precision is about $1.1N$, when the parameter is $N$. In particular, for $N = 14$ standard double precision gives reasonable results. Since constants $\omega_k$ do not depend on $\tau$ they can be tabulated for the values of $n$ that are commonly used in computational finance (e.g., 12 or 14).

The precision requirement is driven by the coefficients $\omega_k$ in (31). Such a high level of precision is not required for the computation of the transform $\tilde{f}$.

## 3.2 The Fast Wiener-Hopf factorization method

We briefly review the framework proposed by [28]. The main contribution of the FWHF–method is an efficient numerical realization of EPV-operators $\mathcal{E}$, $\mathcal{E}^+$ and $\mathcal{E}^-$.

Recall that we consider the procedure for approximations of the Wiener-Hopf factors for the symbol $q/(q+\psi(\xi))$ with $\psi$ being characteristic exponent of RLPE of order $\nu \in (0;2]$ and exponential type $[\lambda_-;\lambda_+]$. The first ingredient is the reduction of the factorization problems to symbols of order 0, which stabilize at infinity to some constant.

Introduce functions

$$\Lambda_-(\xi) = \lambda_+^{\nu_+}(\lambda_+ + i\xi)^{-\nu_+}; \tag{32}$$

$$\Lambda_+(\xi) = (-\lambda_-)^{\nu_-}(-\lambda_- - i\xi)^{-\nu_-}; \tag{33}$$

$$\Phi(\xi) = q\Big((q+\psi(\xi))\Lambda_+(\xi)\Lambda_-(\xi)\Big)^{-1}. \tag{34}$$

Choices of $\nu_+$ and $\nu_-$ depend on properties of $\psi$, hence on order $\nu$ (see (10)–(11)) and drift $\mu$. See details in [28]. First, approximate $\Phi$ by a periodic function with a large period $2\pi/h$, which is the length of the truncated region in the frequency domain, then approximate the latter by a partial sum of the Fourier series, and, finally, use the factorization of the latter instead of the exact one.

Explicit formulae for approximations of $\phi^{\pm}$ have the following form. For small positive $h$ and large even $M$, set

$$b_k^h = \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} \ln\Phi(\xi)e^{-i\xi kh}d\xi, \qquad k \neq 0, \tag{35}$$

$$b_{h,M}^+(\xi) = \sum_{k=1}^{M/2} b_k^h(\exp(i\xi kh)-1), \qquad b_{h,M}^-(\xi) = \sum_{k=-M/2+1}^{-1} b_k^h(\exp(i\xi kh)-1); \tag{36}$$

$$\Phi^{\pm}(\xi) \approx \exp(b_{h,M}^{\pm}(\xi)), \qquad \phi_q^{\pm}(\xi) = \Lambda_{\pm}(\xi)\Phi^{\pm}(\xi). \tag{37}$$

The computational complexity of the Fast Fourier Transform based realization of (35)-(37) is $O(M\ln M)$, where $M$ is a number of points.

We can apply this realization both after the reduction to symbols of order 0 has been made, and without this reduction. In the latter case, $\Lambda_{\pm} = 1$, and we obtain a Poisson type approximation.

It is well-known that the limit of a sequence of the Poisson type characteristic functions is infinitely divisible characteristic function. The converse is also true. Every infinitely divisible characteristic function can be written as the limit of a sequence of finite products of Poisson type characteristic functions. Since $\psi(\xi)$ is the characteristic exponent of Lévy process, then the function $q/(q+\psi(\xi))$ is infinitely divisible characteristic function.

### 3.3 Alternative approximate factorization formulae

If a Lévy process $X_t$ belongs to the class RLPE, then the following integral representations for $\phi_q^+(\xi)$, $\phi_q^-(\xi)$ are valid (see details in [7]):

$$\phi_q^+(\xi) = \exp\left[(2\pi i)^{-1}\int_{-\infty+i\omega_-}^{+\infty+i\omega_-}\frac{\xi\ln(q+\psi(\eta))}{\eta(\xi-\eta)}d\eta\right]; \tag{38}$$

$$\phi_q^-(\xi) = \exp\left[-(2\pi i)^{-1}\int_{-\infty+i\omega_+}^{+\infty+i\omega_+}\frac{\xi\ln(q+\psi(\eta))}{\eta(\xi-\eta)}d\eta\right], \tag{39}$$

where $\omega_- < 0 < \omega_+$ with $\omega_-, \omega_+$ depending on the Lévy process $X_t$ parameters. Notice that the direct computation of $\phi_q^+(\xi)$ and $\phi_q^-(\xi)$ require $O(NM)$ operations, where $N$ is a number of $\xi$-points and $M$ is a number of points for numerical integration in (38)-(39). It makes these formulae very expensive from computational point of view (compare with (35)-(37)).

Below we suggest new formulae for $\phi_q^+(\xi)$, $\phi_q^-(\xi)$ which will improve the computational complexity.

**Theorem 3.1** *Let a Lévy process $X_t$ belongs to the class RLPE. Then there exist constants $\omega_-, \omega_+$, $\omega_- < 0 < \omega_+$ such that*

*a) $\phi_q^+(\xi)$ admits analytical continuation into half-plane $\Im\xi > \omega_-$ and can be represented as follows:*

$$\phi_q^+(\xi) = \exp\left[i\xi F^+(0) - \xi^2\hat{F}^+(\xi)\right], \tag{40}$$

$$F^+(x) = \mathbf{1}_{(-\infty,0]}(x)(2\pi)^{-1}\int_{-\infty+i\omega_-}^{+\infty+i\omega_-} e^{ix\eta}\frac{\ln(q+\psi(\eta))}{\eta^2}d\eta; \tag{41}$$

$$\hat{F}^+(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}F^+(x)dx. \tag{42}$$

*b) $\phi_q^-(\xi)$ admits analytical continuation into half-plane $\Im\xi < \omega_+$ and can be represented as follows:*

$$\phi_q^-(\xi) = \exp\left[-i\xi F^-(0) - \xi^2\hat{F}^-(\xi)\right], \tag{43}$$

$$F^-(x) = \mathbf{1}_{[0,+\infty)}(x)(2\pi)^{-1}\int_{-\infty+i\omega_+}^{+\infty+i\omega_+} e^{ix\eta}\frac{\ln(q+\psi(\eta))}{\eta^2}d\eta; \tag{44}$$

$$\hat{F}^-(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}F^-(x)dx. \tag{45}$$

The proof of Theorem 3.1 follows from a simple idea to represent the integrand in (38)-(39) as follows:

$$\frac{\xi\ln(q+\psi(\eta))}{\eta(\xi-\eta)} = \frac{\ln(q+\psi(\eta))}{\eta^2}\left(-\xi+\frac{\xi^2}{\xi-\eta}\right).$$

It is easy to see that the formulae in Theorem 3.1 can be efficiently realized by means of the Fast Fourier Transform with the computational complexity $O(M\ln M)$, where $M$ is a number of points.

### 3.4 A FFT-based realization of Wiener-Hopf operators as PDOs

Approximants for EPV-operators can be efficiently computed by using the Fast Fourier Transform (FFT) for real-valued functions. Consider the algorithm of the discrete Fourier transform (DFT) defined by

$$G_l = DFT[g](l) = \sum_{k=0}^{M-1} g_k e^{2\pi ikl/M}, \quad l = 0, ..., M-1. \tag{46}$$

The formula for the inverse DFT which recovers the set of $g_k$'s exactly from $G_l$'s is:

$$g_k = iDFT[G](k) = \frac{1}{M}\sum_{l=0}^{M-1} G_l e^{-2\pi ikl/M}, \quad k = 0, ..., M-1. \tag{47}$$

In our case, the data consist of a real-valued array $\{g_k\}_{k=0}^{M}$. The resulting transform satisfies $G_{M-l} = \bar{G}_l$. Since this complex-valued array has real values $G_0$ and $G_{M/2}$, and $M/2-1$ other independent complex values $G_1, ..., G_{M/2-1}$, then it has the same "degrees of freedom" as the original real data set. In this case, it is efficient to use FFT algorithm for real-valued functions (see [36] for technical details). To distinguish DFT of real functions we will use notation RDFT.

Fix the space step $h > 0$ and number of the space points $M = 2^m$. Define the partitions of the normalized log-price domain $[-\frac{Mh}{2}; \frac{Mh}{2})$ by points $x_k = -\frac{Mh}{2} + kh$, $k = 0, ..., M-1$, and the frequency domain $[-\frac{\pi}{h}; \frac{\pi}{h}]$ by points $\xi_l = \frac{2\pi l}{hM}$, $l = -M/2, ..., M/2$. Then the Fourier transform of a function $g$ on the real line can be approximated as follows:

$$\hat{g}(\xi_l) \approx h e^{i\pi l}\overline{RDFT[g](l)}, \quad l = 0, ..., M/2.$$

Here and below, $\overline{z}$ denotes the complex conjugate of $z$. Using the notation $p(\xi) = q(q+\psi(\xi))^{-1}$, we can approximate $\mathcal{E}_q$:

$$(\mathcal{E}_q g)(x_k) \approx iRDFT[\overline{p}.*RDFT[g]](k),\ k = 0, ..., M-1. \tag{48}$$

Here and below, $.*$ is the element-wise multiplication of arrays that represent the functions. Further, using (35)-(37) or Theorem 3.1 we define $p^{\pm}(\xi_l)$ as approximate values of $phi_q^{\pm}(\xi_l)$, $l = -M/2, ..., 0$. The action of the EPV-operator $\mathcal{E}_q^{\pm}$ is approximated as follows:

$$(\mathcal{E}_q^{\pm} g)(x_k) = iRDFT[\overline{p^{\pm}}.*RDFT[g]](k),\ k = 0, ..., M-1. \tag{49}$$

### 3.5 The Gaver-Stehfest algorithm and the FWHF-method

In our study we apply the Laplace transform to the problem (21) for pricing barrier options under Lévy models. Then we solve the corresponding problem in the Laplace domain at real positive values of the transform parameter specified by the Gaver-Stehfest algorithm.

We introduce a new variable $\tau = T - t$. With a new function $f(\tau, x) = F(T-\tau, x)$ the problem (21)-(23) turns into

$$(\partial_\tau + r - L)f(\tau, x) = 0, \qquad x > 0, \tau > 0, \tag{50}$$

$$f(\tau, x) = 1, \qquad x \le 0, \tau \ge 0, \tag{51}$$

$$f(0, x) = 0, \qquad x > 0, \tag{52}$$

Then we introduce a new function $v(\tau,x) = f(\tau,x) - 1$, and we obtain

$$(\partial_\tau + r - L)v(\tau,x) = -r, \qquad x > 0, \tau > 0, \tag{53}$$
$$v(\tau,x) = 0, \qquad x \le 0, \tau \ge 0, \tag{54}$$
$$v(0,x) = -1, \qquad x > 0. \tag{55}$$

The Laplace transform of $v(\tau,x)$ with respect to the time variable is defined by

$$\tilde{v}(\lambda,x) := \int_0^\infty e^{-\lambda\tau} v(\tau,x)\, d\tau,$$

where $\lambda$ is a transform variable with positive real part, $\Re\lambda > 0$. To be specific, in subsequent study we assume that $\lambda \in \mathbf{R}_+$. The standard rules yield

$$\partial_\tau v(\tau,x) \mapsto \lambda\tilde{v}(\lambda,x) - v(0,x),\;\; Lv(\tau,x) \mapsto L\tilde{v}(\lambda,x).$$

Applying Laplace transform to (53), we obtain that $\tilde{v}(\lambda,x)$ satisfies the following equation:

$$(\lambda + r - L)\tilde{v}(\lambda,x) = -(\lambda + r)\lambda^{-1}, \quad x > 0, \tag{56}$$

subject to the corresponding transformed boundary condition

$$\tilde{v}(\lambda,x) = 0,\;\; x \le 0. \tag{57}$$

Given $n$, we can use the Gaver-Stehfest inversion formula for $\tilde{v}(\lambda,x)$ provided that the solutions to the problem (56),(57) are found at $\lambda = k\ln 2/T$, $k = 1,\dots,N$ (see (28)–(31)).

Set $q = \lambda + r$ and denote by $\mathbf{1}_{(0,+\infty)}(x)$ the indicator function of $(0,+\infty)$. A general class of boundary problems

$$(q - L)u(x) \;=\; g(x), \quad x > 0, \tag{58}$$
$$u(x) \;=\; 0, \quad x \le 0; \tag{59}$$

that contains the problem (56)-(57) was studied in [7]. It was shown that the unique bounded solution to (58)-(59) is given by

$$u(x) = \frac{1}{q}\left(\mathcal{E}_q^- \mathbf{1}_{(0,+\infty)} \mathcal{E}_q^+ g\right)(x). \tag{60}$$

Taking into account that $g(x) = (\lambda + r)\lambda^{-1}$ and simplifying (60), we obtain

$$\tilde{v}(\lambda,x) = -\lambda^{-1}\mathcal{E}_q^- \mathbf{1}_{(0;+\infty)}(x). \tag{61}$$

Now, the Fast Wiener-Hopf factorization method [28] can be applied. Since the approximate expressions for the Wiener-Hopf factors $\phi_q^\pm(\xi)$ are available (see 35), one can calculate $\tilde{v}(\lambda,x)$ quite easily using formulae (49).

It follows, that the computational complexity of the developed algorithm (as well as the FWHF-method) is $O(NM\ln M)$, where $M$ is a number of points in the log-price space; in the case of the FWHF-method, $N$ denotes the number of time steps. The Gaver-Stehfest algorithm produces rapid convergence results already using $N = 10 - 14$ apart from the FWHF-method with $N$ being of order $400 - 800$. Hence, the new method is computationally much faster (often, dozens of times faster) than the original FWHF-method constructed in [28].

Our new method enjoys an additional appealing feature: it produces a set of option prices at different spot levels. Notice that in the case of the known Laplace transform methods, one must perform numerical Laplace inversion separately for each initial spot price of the underlying.

Our new algorithm provides increasing accuracy as $N$ in the Gaver-Stehfest inversion formula increases. However, if $N > 14$ good accuracy results can be achieved only using a multi-precision computational environment.

The method based on the Post-Widder formula (see the next subsection) achieves similar performance to the method proposed here; however, the former method does not require high precision.

**Remark 3.2** *The problem (56),(57) can be also solved by a finite difference method (e.g. [13, 33, 30]). Since the number of time steps in finite difference schemes is sufficiently large (several hundreds, or thousands), we can significantly improve the speed of such methods.*

## 3.6 The Post-Widder formula or Carr's randomization

In this subsection, we propose the second new approach to pricing barrier options which involves the numerical Laplace transform inversion formulae (26), (27). Recall that we are looking for the solution $v(\tau, x)$ to the problem (53)-(54) at $\tau = T$.

Applying the Laplace transform to the corresponding PIDE, we consider the problem (56), (57) in the Laplace domain, once again. As a basis for the Gaver-Stehfest algorithm, it was established a discrete analog of the Post-Widder formula (24) involving finite differences to approximate $N$th derivative of the transformed function. In fact, for performing numerical inversion we need to find $\partial_\lambda^N \tilde{v}(\lambda, x)$.

We have, on differentiating both sides of the equations (56),(57) with respect to $\lambda$:

$$(\lambda + r - L)\partial_\lambda \tilde{v}(\lambda, x) = -\tilde{v}(\lambda, x) + \frac{r}{\lambda^2}, x > 0, \tag{62}$$

$$\partial_\lambda \tilde{v}(\lambda, x) = 0, \; x \le 0. \tag{63}$$

Repeating this procedure, for all $k = 1, 2, ..., N-1$, we obtain a sequence of the following problems

$$(\lambda + r - L)\partial_\lambda^k \tilde{v}(\lambda, x) = -k\partial_\lambda^{k-1}\tilde{v}(\lambda, x) + \frac{(-1)^{k+1} r}{k!\lambda^{k+1}}, x > 0, \tag{64}$$

$$\partial_\lambda \tilde{v}(\lambda, x) = 0, \; x \le 0. \tag{65}$$

Fix an integer $N > 1$, and set $\Delta\tau = T/N$, $\lambda = 1/\Delta\tau$. Then we introduce the following functions:

$$v_0(x) = -\mathbf{1}_{(0,+\infty)}(x); \tag{66}$$

$$v_{k+1}(x) = \frac{(-1)^k}{k!}\left(\frac{1}{\Delta\tau}\right)^{k+1} \partial_\lambda^k \tilde{v}\left(\frac{1}{\Delta\tau}, x\right), \; k = 0, ..., N-1. \tag{67}$$

It follows that

$$\partial_\lambda^k \tilde{v}\left(\frac{1}{\Delta\tau}, x\right) = (-1)^k k! (\Delta\tau)^{k+1} v_{k+1}(x), k = 0, ..., N-1. \tag{68}$$

Substituting expressions $1/\Delta\tau$ for $\lambda$ and (68) for $\partial_\lambda^k \tilde{v}\left(\frac{1}{\Delta\tau}, x\right)$ into (64)-(65), simplifying and eliminating the multipliers from the final set of equations, one finds for $k = 1, ..., N$:

$$(q - L)v_k(x) = \frac{1}{\Delta\tau} v_{k-1}(x) - r, \quad x > 0, \tag{69}$$

$$v_k(x) = 0, \quad x \le 0, \tag{70}$$

where $q = r + 1/\Delta\tau$.

The sequence $v_k(x)$, $k = 1, 2, ..., N+1$, is determined recurrently by means of the problem (69), (70) at each step $k$. It follows from [7] that the unique bounded solution to the problem (69), (70) is given by

$$v_k = \frac{1}{q\Delta\tau}\mathcal{E}_q^- \mathbf{1}_{(0,+\infty)}\mathcal{E}_q^+(v_{k-1} - r\Delta\tau). \tag{71}$$

Once again, the Fast Wiener-Hopf factorization method [28] can be applied. Moreover, the only one approximate formula for Wiener-Hopf operator $\mathcal{E}_q^-$ (49) is needed at the last step. At all intermediate steps, the exact analytic expression $q/(q + \psi(\xi)$ is used (see (48)). Indeed, set

$$w_1 = \mathbf{1}_{(0;+\infty)}\mathcal{E}_q^+(v_0 - r\Delta\tau) = \mathbf{1}_{(0;+\infty)}(-1 - r\Delta\tau). \tag{72}$$

For $k = 2, \dots, N$, define

$$w_k = \mathbf{1}_{(0;+\infty)}\mathcal{E}_q^+(v_{k-1} - r\Delta\tau). \tag{73}$$

Then

$$v_k = (q\Delta\tau)^{-1}\mathcal{E}_q^- w_k. \tag{74}$$

Using the Wiener-Hopf factorization formula (19), we obtain that for $k = 2, \dots, N$

$$w_k = \mathbf{1}_{(0;+\infty)}(x)((q\Delta\tau)^{-1}\mathcal{E}_q w_{k-1}(x) - r\Delta\tau). \tag{75}$$

Since we are looking for $v_N(x)$, we need to apply the formula (74) at the step $k = N$ only.

Finally, we take into account the Post-Widder formula (24)-(25). As a result, we conjecture that the solution $v_N(x)$ to our problem converges to the unknown solution $v(T, x)$ of the problem (53)-(54), as $N$ gets arbitrarily large with $T$ held fixed.

Unfortunately, the Post-Widder formula provide a very poor approximation (of order $N^{-1}$). See details in Subsection 3.1. For example, $v_{1000}(x)$ may yield an estimate to $v(T, x)$ with only two or three digits of accuracy. To achieve a good approximation, a convergence acceleration algorithm is required for the sequence $v_N(x)$. A good candidate is the summation formula (26)-(27) (see [1]). We start with the choice $N = 5$ and $m = 3$, and increase them if necessary.

Given parameters $N$ and $m$ in (26)-(27), the computational complexity of the developed algorithm is $O(N_0 M \ln M)$, where $M$ is a number of points in the log-price space, and $N_0 = \frac{N(m+1)m}{2}$.

The new enhanced FWHF-method based on the Post-Widder formula produces rapid convergence results already using $N = 10$ and $m = 3$. Hence, the new method is computationally much faster than the original FWHF-method developed in [28].

The second new method achieves similar performance to the first one constructed in the previous Subsection. Our new algorithm provides increasing accuracy as $N$ and $m$ in the formula (26) increase. At the same time, the method does not require a multi-precision arithmetic.

**Remark 3.3** *Notice that our value $v_k(x)$ is also the approximation for the solution $v(k\Delta\tau, x)$ to the problem (53)-(54) which arises when time is discretized and the derivative $\partial_\tau v(k\Delta\tau, x)$ in (53) is replaced with the finite difference:*

$$(1/\Delta\tau)(v(k\Delta\tau, x) - v((k-1)\Delta\tau, x)).$$

*The notion of discretizing time while leaving space continuous is known in the numerical methods literature as the method of horizontal lines or Rothe's method [38]. Carr's randomization procedure [9] indicates an alternative probabilistic interpretation of the approximation induced by our procedure. Notice that Carr's randomization was successfully applied to the valuation of (single and double) barrier options in a number of works.*

Taking into account properties of the Post-Widder functionals (25) (see the corresponding results in [20, 21]) and the method of lines interpretation obtained we conclude that the following proposition holds.

**Proposition 3.4** *Let $X_t$ be a RLPE with an infinitesimal generator (7), $v(\tau, x)$ – solution to the problem (53)-(55); $v_j^{(N)}(T,x)$, $j = N-1, N-2, ..., 0$ – consequence of the solutions to the problems (64)–(65) with $v_0^{(N)}(T,x) = -\mathbf{1}_{(0;+\infty)}(x)$, $q = \Delta\tau^{-1} + r$ and $\Delta\tau = T/N$. Then for each fixed $x$ the following properties hold:*

- *the upper and lower bounds are $-1 \le v_N^{(N)}(T,x) \le 0$, $N > 1$;*
- *the number of roots of the equation $v_N^{(N)}(T,x) = C$ with respect to $T$ does not exceed the number of roots of the equation $v(T,x) = C$, $N > 1$, $C > 0$;*
- *the following asymptotic formulae are valid:*

$$v_N^{(N)}(T,x) - v(T,x) \sim \sum_{j=1}^{\infty} \frac{c_j(T,x)}{N^j}, N \to \infty \tag{76}$$

$$\sum_{k=1}^{m} w(k,m) v_{Nk}^{(Nk)}(T,x) - v(T,x) = O(N^{-m}), N \to \infty \tag{77}$$

*where $m \in \mathbf{N}$, $w(k,m)$ are defined by (27).*

**Remark 3.5** *As a result, we conjecture that the method of lines (or Carr's) approximation to the value of a finite-lived first touch digital option always converges to the actual value for a wide class of Lévy processes. Moreover, we provide the order of the convergence: $O(N^{-1})$, where $N$ is the number of time steps.*

Notice that in the case of first touch digital options, the Fourier integral (9) for the function $v_0$ diverges as $\xi \in \mathbf{R}$. In order to apply our Fourier transform based approach we need to use a trick with weight functions.

Let a characteristic exponent $\psi$ of Lévy process $X_t$ is holomorphic in $\Im\xi \in (\lambda_-; \lambda_+)$, $\lambda_- < -1 < 0 < \lambda_+$. The condition holds for RLPE, in particular. Fix a real number $\omega$, $\lambda_- < \omega < 0$, and introduce functions $w_j^\omega(x) = w_j(x)e^{\omega x}$. Set $q_\omega = q + \psi(i\omega)$ and consider the operator $\mathcal{E}_{q_\omega}$ with symbol $q_\omega(q_\omega + \psi_1(\xi))^{-1}$, where $\psi_1(\xi) = \psi(\xi + i\omega) - \psi(i\omega)$. It is easy to check that the following relation is valid.

$$\phi_q^-(i\omega)\mathcal{E}_{q_\omega}^- = e^{\omega x}\mathcal{E}_q^- e^{-\omega x}, \qquad \phi_q^+(i\omega)\mathcal{E}_{q_\omega}^+ = e^{\omega x}\mathcal{E}_q^+ e^{-\omega x}, \tag{78}$$

$$q\mathcal{E}_{q_\omega} = e^{\omega x}\mathcal{E}_q e^{-\omega x} q_\omega. \tag{79}$$

Taking into account (79), the formula (75) can be rewritten in terms of the functions $w_j^\omega(x)$ as follows:

$$w_j^\omega(x) = \mathbf{1}_{(0;+\infty)}(x)((q_\omega\Delta\tau)^{-1}\mathcal{E}_{q_\omega} w_{j-1}^\omega(x) - r\Delta\tau e^{\omega x}). \tag{80}$$

The modified algorithm includes the following steps:

- find $w_1^\omega(x)$ using (72):
$$w_1^\omega(x) = e^{\omega x}\mathbf{1}_{(0;+\infty)}(x)(-1 - r\Delta\tau);$$
- $j = 2, \dots, N$, find $w_j^\omega(x)$:

- define $v_N(x)$ by the formula (see (78) and (74)):

$$v_N(x) = (q_\omega \Delta\tau)^{-1} e^{-\omega x} \phi_q^-(i\omega) \mathcal{E}_{q_\omega}^- w_N^\omega(x),$$

where $\phi_q^-(i\omega)$ can be find by the formula (39).

# 4 Numerical examples

In this section, we compare the performance of the new two methods and the original FWHF–method. In numerical examples, we implement the algorithms of the enhanced FWHF–methods described in Subsection 3.5 and in Subsection 3.6. We will refer to these algorithms as the FWHF&GS-method and FWHF&PW-method, repectively. The valuse were calculated on a PC with characteristics Intel Core(TM)I5 CPU, 1.8GHz, RAM 4Gb, under Windows 8.1.

We will show the advantage of the new methods in terms of speed over the original FWHF-method. The two examples, which we analyze in detail below, are fairly representative. The localization domain in both examples is $(x_{\min}; x_{\max})$ with $x_{\min} = -\ln 2$ and $x_{\max} = \ln 2$; we check separately that if we increase the domain two-fold, and the number of points 4-fold, the prices change by less than 0.0001.

Firstly, we will compare the first passage probabilities from the FWHF&GS-method and the FWHF&PW-method against probabilities obtained using explicit formulae for the Wiener-Hopf factors under Laplace transform and reported in [26]. Notice that the method in [26] requires a high precision arithmetic, and we will use the probabilities from Table 1 and Table 2 in [26] as the benchmarks. Table 1 reports the first passage probabilities $P[T' \leq T]$ under Kou model with parameters: $\lambda_+ = 33.33$, $\lambda_- = -50$, $c_+ = 1.5$, $c_- = 1.5$ for two drift levels: positive ($\mu = 0.1$) and negative ($\mu = -0.1$). The initial value of a stock $S_0 = 1$, the barrier $H = exp(0.3) \approx 1.349858808$, and time $T = 1$. The parameters are taken from [26] with corespondent reparametrization (see Example 2). Notice that in [26], $T'$ is a first entrance of $S_t$into $[H, +\infty)$. The results from the FWHF&PW and FWHF&GS methods converge very fast and agree with the probabilities in [26].

As a second example, we consider the first touch digital option with barrier $H$ from below and time to expiry $T$ in KoBoL (CGMY) model of order $\nu \in (0, 1)$, The parameters of the model are $\sigma = 0$, $\nu = 0.5$, $\lambda_+ = 9$, $\lambda_- = -8$, $c = 1$. We choose instantaneous interest rate $r = 0.072310$, time to expiry $T = 0.5$ year, and the barrier $H = 90$. In this case, the drift parameter $\mu$ is approximately zero. Table 2, reports prices for first touch digital options calculated by using the FWHF method with very fine grids, and FWHF&PW and FWHF&GS methods. The options are priced at the spot level $S = 100$. The prices from the FWHF&PW and FWHF&GS methods converge very fast and agree with the prices obtained by the original FWHF.

We see that FWHF&PW and FWHF&GS methods in both examples produce sufficiently good results in just several dozens of milliseconds.

# 5 Conclusion

In the paper, we show the advantages of the Laplace transform-based approach in the context of the pricing first touch digital options in exponential Lévy models. In particular, we show that the computational efficiency of the numerical methods which start with the time

Table 1: Kou model: first passage probabilities $P[T' \leq T]$

**A**

| The method | KW | FWHF&PW | | FWHF&GS | | FWHF |
|---|---|---|---|---|---|---|
| Parameters | $N = 10$ | $h = 0.0001$ $N = 5$, $m = 3$ | $h = 0.001$ $N = 5$, $m = 3$ | $h = 0.0001$ $N = 14$ | $h = 0.001$ $N = 14$ | $h = 0.0001$ $N = 2000$ |
| $\mu = 0.1$ | 0.25584 | 0.25579 | 0.25522 | 0.25591 | 0.25673 | 0.25573 |
| $\mu = -0.1$ | 0.06122 | 0.06121 | 0.06098 | 0.06125 | 0.06158 | 0.06119 |
| CPU-time (sec) | | 0.29 | 0.029 | 0.39 | 0.039 | 11.3 |

Kou model parameters: $\lambda_+ = 33.33$, $\lambda_- = -50$, $c_+ = 1.5$, $c_- = 1.5$.
Option parameters: $S_0 = 1$, $H = exp(0.3) \approx 1.349858808$, $T = 1$.
Algorithm parameters: $h$ – space step, $N$ – number of time steps (or the parameter of the FWHF&PW and FWHF&GS methods), $m$ – number of terms in the acceleration formula

Table 2: KoBoL(CGMY) model: option prices

| | FWHF | FWHF&PW | | | FWHF&GS | |
|---|---|---|---|---|---|---|
| Spot price | $h = 0.0001$ $N = 1600$ | $h = 0.001$ $N = 5$ | $h = 0.001$ $N = 10$ | $h = 0.0001$ $N = 10$ | $h = 0.001$ $N = 14$ | $h = 0.0001$ $N = 14$ |
| $S = 100$ | 0.36626 | 0.36430 | 0.36435 | 0.36629 | 0.36558 | 0.36426 |
| CPU-time (sec) | 4.2 | 0.015 | 0.024 | 0.21 | 0.025 | 0.22 |

KoBoL parameters: $\nu = 0.5$, $\lambda_+ = 9$, $\lambda_- = -8$, $c = 1$, $\mu \approx 0$.
Option parameters: $H = 90$, $r = 0.072310$, $T = 0.5$.
Algorithm parameters: $h$ – space step, $N$ – number of time steps (or the parameter of the FWHF&PW and FWHF&GS methods), $S$ – spot price.

discretization can be significantly enhanced (often, in several dozen of times) by means of the Laplace transform technique.

Moreover, we propose two new fast and accurate methods for pricing barrier options in wide classes of Lévy processes. Both methods use the numerical Laplace transform inversion formulae and the Fast Wiener-Hopf factorization method developed in [28]. The first method uses the Gaver-Stehfest algorithm, the second one – the Post-Widder formula. In the present paper we also suggest new Wiener-Hopf factorization formulae which can be also efficiently implemented into the methods.

Numerical examples show that the new methods are computationally much faster (often, dozen of times faster) than the original FWHF-method constructed in [28]. Our new methods enjoy an additional appealing feature: they produce a set of option prices at different spot levels, simultaneously.

The method based on the Post-Widder formula achieves similar performance to the method which uses the Gaver-Stehfest algorithm; however, the former method does not require high precision.

We notice that Carr's randomization procedure in [9] indicates an alternative interpretation of the approximation induced by our second method. As a result, we conjecture that Carr's approximation to the value of a finite-lived digital option always converges to the actual value for a wide class of Lévy processes. Moreover, we provide the order of the convergence and some properties of the approximate prices.

The framework proposed in the paper can be extended to regime switching Lévy models. The method based on the Post-Widder formula under regime switching in Lévy-driven models was implemented into the program platform Premia (www.premia.fr). Premia is a software designed for option pricing, hedging and financial model calibration.

**Acknowledgements**

The author is grateful to participants of The International Workshop "Wiener-Hopf method, Toeplitz operators and their application", Veracruz, Mexico, November 3-7, 2015, and especially to Professor Sergei Grudsky for discussions and useful comments.

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